Fibonacci Numbers

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PROMYS 2022

Fibonacci numbers are sequences of whole numbers arranged in such a way that every number is the sum of the preceding two numbers. Each number is represented by F_n in the sequence.

The First 24 Fibonacci Numbers are :

n	F_n	n	F_n	n	F_n	n	F_n
0	0	6	8	12	144	18	2584
1	1	7	13	13	233	19	4181
2	1	8	21	14	377	20	6765
3	2	9	34	15	610	21	10946
4	3	10	55	16	987	22	17711
5	5	11	89	17	1597	23	28657

The Fibonacci numbers can be calculated by using this formula $F_n = F_{n-1} + F_{n-2}$ for $n \geq 1$

Calculating the first few values of the Fibonacci Numbers:

n	$F_{n-1} + F_{n-2}$	F_n
0		0
1		1
2	1+0	1
3	1+2	3
4	2+3	5
5	3+5	8

Before we move further with exploration, let's define one of the most important and frequently used term in our whole research.

Period - Let (F_n) be Fibonacci sequence, we say (F_n) is p-periodic in mod m if :

$F_n = F_{n+p}$ (mod m) $\forall\ n \in N$

We are particularly interested in the least period $\pi(m)$ because p is not unique.

Now, What happens if we reduce to modulo 5?

$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657\}$
$\rightarrow$ $(0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, 0\}$.

Observation - Notice that the pattern 1011 from which we started repeats after the 20th term. This implies that the Fibonacci numbers reduced to modulo 5, have a period of 20. This also shows an example of calculating a period.

Now, we are trying to find $\pi(m)$ for all m and see if there is any formula. We can begin by going through numericals and making a table, and using code to speed up the process. The code is:

```
lst =[1,1]
m=(insert m here)
count = 2
for i in range(2,1000):
    if (lst[i-1]+lst[i-2]) < m:
        lst.append(lst[i-1]+lst[i-2])
    if (lst[i-1]+lst[i-2]) >= m:
        lst.append(lst[i-1] + lst[i-2] - m)
    count += 1
    if (lst[i] == 1) and (lst[i-1] == 2):
        break
print(count-1)
```

The first line declares the sequence, in which we are planning to write out the sequence of the Fibonacci numbers modulo m. The next 5 lines calculate the next term from the previous two terms, and append the new term onto the list. The last 3 lines go through the code, and determine if there has been a repetition, and then outputs the count, which is the numbers of term in this sequence before the repetition, which by definition, is the period.

Trying this for the first few values of m yields the table:

m	$\pi_L(m)$	m	$\pi_L(m)$	m	$\pi_L(m)$	m	$\pi_L(m)$
1	1	9	24	17	36	25	100
2	3	10	60	18	24	26	84
3	8	11	10	19	18	27	72
4	6	12	24	20	60	28	48
5	20	13	28	21	16	29	14
6	24	14	48	22	30	30	120
7	16	15	40	23	48	31	30
8	12	16	24	24	24	32	48

The main objective of this write-up is to find $\pi(m)$ from m.

We can split up all m into three cases:

1. m is prime.
2. m is a prime power
3. m is has multiple prime factors.

First case in where m is prime: We can begin by going through some numericals. Using the code to test some prime values of m yields the table:

m	$\pi(m)$	m	$\pi(m)$
2	3	23	48
3	8	29	14
5	20	31	30
7	16	37	76
11	10	41	40
13	28	43	88
17	36	47	32
19	18	53	108

Some immediate conjectures that we came up off the bat were that if $m \equiv 3 \mod 10$, then $\pi(m) = 2(p+1)$. This is because $\pi(3) = 8$, $\pi(13) = 28$, $\pi(23) = 48$, etc. However, when we get to $m = 113$, then we have

$\pi(113) = 76$, which is a counterexample to our conjecture, since $2(113+1) = 228 \neq 76$. However, note that $76 \mid 228$, so we salvaged our conjecture into if $m \equiv 3 \mod 10$, then $\pi(m) \mid 2(p+1)$. We have not been able to find a counterexample to this, however, we have also not been able prove it, so it remains a conjecture. After further investigation, we found that not only did this hold true for $m \equiv 3 \mod 10$, but also for all $m \equiv \pm 2 \mod 5$.

Another similar conjecture that we came up right off the bat was that if $m \equiv 1 \mod 10$, then $\pi(m) = p - 1$. This is because $\pi(11) = 10$, $\pi(31) = 30$, $\pi(41) = 40$, etc. However, when we get to $m = 101$, then we have $\pi(101) = 50$, which is a counterexample to our conjecture, since $101 - 1 = 100 \neq 50$. However, note that $50 \mid 100$, so we salvaged our conjecture into if $m \equiv 1 \mod 10$, then $\pi(m) \mid \pm p - 1$. Similar to above, we haven't been able to find a counterexample to this statement, however, at the same time, we haven't been able to prove it, so it remains a conjecture. After further investigation, similar to above, we found that not only did this hold true for $m \equiv 1 \mod 10$, we have that it holds true for all $m \equiv \pm 1 \mod 5$.

We have not been able to notice anything else for primes, so this is about as far as we have gotten in finding $\pi(m)$ when m is prime.

Now, onto the second case in which m is a prime power which can be written as a p^α, where p is a prime and α is a positive integer. We can begin by doing numericals to find $\pi(m)$ that fit this definition. First, when $p = 2$ yields the following table:

n	2^n	$\pi(2^n)$
1	2	3
2	4	6
3	8	12
4	16	24
5	32	48

Notice that the period of two consecutive powers of 2 differ by a factor of 2. So, we can conjecture that

$$\pi(2^{n+1}) = 2 \cdot \pi(2^n) \text{ where } n \geq 1$$

.

We also have that $\pi(2^n) = 2 \cdot \pi(2^{n-1}) \implies \pi(2^{n+1}) = 2^2 \cdot \pi(2^{n-1})$. Continuing this up until we reach $\pi(p)$, yields that $\pi(2^{n+1}) = 2^n \cdot \pi(2)$. So, we can convert the first conjecture into the new conjecture:

$$\pi(2^n) = 2^{n-1} \cdot \pi(2).$$

Now, let's consider some other numericals with different bases p. Trying $p = 3, 5, 7$ yield the following tables:

n	3^n	$\pi(3^n)$
1	3	8
2	9	24
3	27	72
4	81	216
5	243	648

n	5^n	$\pi(5^n)$
1	5	20
2	25	100
3	125	500
4	625	2500
5	3125	12500

n	7^n	$\pi(7^n)$
1	7	16
2	49	112
3	343	784
4	2401	5488
5	16807	38416

We can see that the same pattern applies, in which the period of two consecutive powers of p differ by a factor of p. So, we can generalize our conjecture from $p = 2$ into p, yielding:

$$\pi(p^n) = p^{n-1} \cdot \pi(p) \text{ where } n \geq 1$$

We can see that the only necessary number to calculate the period of a prime power is the period of the prime itself.

Now, consider the third case in which m has multiple prime factors. Let $m = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$.

First, using the table of periods for the Fibonacci numbers above, we found the property where $a \mid b \implies \pi(a) \mid \pi(b)$. This is because the sequence of Fibonacci numbers modulo b can be written as $a \cdot k$ where $k \in \mathbb{Z}$, since $a \mid b$, so for b to repeat, a must also begin its repetition, which by definition, has to be a multiple of $\pi(a)$. So, we have that $\pi(b)$ is a multiple of $\pi(a)$, so we have $\pi(a) \mid \pi(b)$.

We also found the property that $\pi(m)$ was the lowest common multiple of the period of all prime components of m. For example, $\pi(20) = [\pi(4), \pi(5)]$.

Theorem: If $m = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, then $\pi(m) = [\pi(p_1^{n_1}), \pi(p_2^{n_2}), \dots, \pi(p_k^{n_k})]$.

Proof. By $a \mid b \implies \pi(a) \mid \pi(b)$, we have that since $p_1^{n_1} \mid m \implies \pi(p_1^{n_1}) \mid \pi(m)$.

Similarly, we have that $p_2^{n_2} \mid m \implies \pi(p_2^{n_2}) \mid \pi(m)$ all the way up to $\pi(p_k^{n_k}) \mid \pi(m)$.

So, $\pi(m)$ must be a multiple of $\pi(p_1^{n_1})$, a multiple of $\pi(p_2^{n_2})$, ..., a multiple of $\pi(p_k^{n_k})$.

In other words, $\pi(m)$ is a common multiple of $\pi(p_1^{n_1})$, $\pi(p_1^{n_2})$, ..., $\pi(p_k^{n_k})$.

By the definition of period that we had earlier, we only care about the least number that fits this property. Therefore, since $\pi(m)$ must be a common divisor, $\pi(m)$ must be the least common divisor of all the prime components. QED.

Note that all of the prime components are in the form of prime powers, which was case 2, so we can find the period of all numbers with multiple factors from the period of prime powers, which can be found from the period of primes.

□

Now that we have explored some patterns for the period of Fibonacci Numbers, let's take a look at some other sequences defined by linear recursion. The next simplest sequence is probably the Lucas Numbers. They are also defined by the recursion $L_n = L_{n-1} + L_{n-2}$, however, they have the starting values $L_0 = 1$ and $L_1 = 3$. Calculating the first few values of the Lucas Numbers:

n	$L_{n-1} + L_{n-2}$	L_n	n	$L_{n-1} + L_{n-2}$	L_n
0		1	5	7+11	18
1		3	6	11+18	29
2	1+3	4	7	18+29	47
3	3+4	7	8	29+47	76
4	4+7	11	9	47+76	123

Now, let's take a look when we take modulo 3 of the Lucas Number sequence.

$$\{1, 3, 4, 7, 11, 18, 29, 47, 76, 123...\} \rightarrow \{(1, 0, 1, 1, 2, 0, 2, 2), 1, 0...\}$$

.

Notice that this sequence repeats after the 8th term, so its period is 8. The code we used to determine the period of Lucas numbers modulo m is very similar to the code to find the period of Fibonacci numbers. All we had to do was change the beginning of the list into $(1, 3)$ instead of $(1, 1)$:

```
lst =[1 ,3]
m=7
count = 2
for i in range(2 ,1000):
    if (lst[i-1]+lst[i-2]) < m:
        lst.append(lst[i-1]+lst[i-2])
    if (lst[i-1]+lst[i-2]) >= m:
        lst.append(lst[i-1] + lst[i-2] - m)
    count += 1
    if (lst[i] == 1) and (lst[i-1] == 2):
        break
print(count-1)
```

Now, after using this code to find the periods of different m, we have the table, where $\pi_L(m)$ is the period of the Lucas Numbers modulo m:

m	$\pi_L(m)$	m	$\pi_L(m)$	m	$\pi_L(m)$	m	$\pi_L(m)$
1	1	9	24	17	36	25	20
2	3	10	12	18	24	26	84
3	8	11	10	19	18	27	72
4	6	12	24	20	12	28	48
5	4	13	28	21	16	29	14
6	24	14	48	22	30	30	24
7	16	15	8	23	48	31	30
8	12	16	24	24	24	32	48

We can see that the two properties of the period for non-prime numbers hold the same. For example, $\pi_L(16) = 8 \cdot \pi_L(2)$ and $\pi_L(25) = 5 \cdot \pi_{:}(5)$. Also, $\pi_L(20) = [\pi_L(4), \pi_l(5)]$.

Comparing this with the table of periods for Fibonacci numbers, we can see that $\pi(m) = \pi_L(m)$, except for $m \equiv 0 \mod 5$. In that case $\pi(m) = 5 \cdot \pi_L(m)$. Note that $\pi(5) = 20$, $\pi_L(5) = 4$, which implies $\pi(5) = 5 \cdot \pi_L(5) \implies \pi(5^n) = 5 \cdot \pi_L(5)$. Using our method of finding the period using the LCM, we can see that the latter statement is a corollary. The former statement that states $\pi(m) = \pi_L(m)$, when $m \not\equiv 0 \mod 5$ remains a conjecture.

Other than the Lucas Numbers, some other numbers defined by Linear Recursion include the Pell numbers, the Tribonacci numbers, and the Jacobsthal numbers, which we will explore.

1. Pell numbers: The Pell numbers are defined by the linear recursion $P_n = 2P_{n-1} + P_{n-2}$, with $P_0 = 0$ and $P_1 = 1$.

n	2P$_{n-1} + P_{n-2}$	P$_n$	n	2P$_{n-1} + P_{n-2}$	P$_n$
0		0	5	2(12)+5	29
1		1	6	2(29)+12	70
2	2(1)+0	2	7	2(70)+29	169
3	2(2)+1	5	8	2(169)+70	408
4	2(2)+2	12	9	2(408)+169	985

Now, let's take a look when we take modulo 3 of the Pell Number sequence.

$$\{1, 2, 5, 12, 29, 70, 169, 408, 985, \dots\} \to \{(1, 2, 0, 2, 2, 1, 0, 1), 1, 2...\}$$

.

Notice that this sequence repeats after the 8th term, so its period is 8. We can use similar code to what we used for the Fibonacci and Pell numbers to find the periods of the Pell numbers. Doing that for the first 40 m yields the following table:

m	$\pi_p(m)$	m	$\pi_p(m)$	m	$\pi_p(m)$	m	$\pi_p(m)$	m	$\pi_j(m)$
1	1	9	24	17	16	25	60	33	24
2	2	10	12	18	24	26	28	34	16
3	8	11	24	19	40	27	72	35	12
4	4	12	8	20	12	28	12	36	24
5	12	13	28	21	24	29	20	37	76
6	8	14	6	22	24	30	24	38	40
7	6	15	24	23	22	31	30	39	56
8	8	16	16	24	8	32	32	40	24

Examining the table, we see that the two conjectures for finding the period of Fibonacci sequence modulo non-prime m also hold true for the Pell numbers. For example, we have that $\pi_P(16) = 16 = 8 \cdot 2 = 8 \cdot \pi_P(2)$. We also have the other conjecture holding true, for example: $\pi_P(20) = 12 = [4, 12] = [\pi_P(4), \pi_P(5)]$. The proof should be the exact same to the proof for Fibonacci.

2. Tribonacci Numbers

The Tribonacci series is an extended version of the Fibonacci sequence where each term is the sum of the three preceding terms as opposed to the Fibonacci sequence where take the sum of the proceeding two terms.

The Tribonacci Numbers are defined by the linear recursion $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, with $T_0 = 0$, $T_1 = 1$, and $T_2 = 1$.

n	T$_{n-1} + T_{n-2} + T_{n-3}$	T$_n$	n	T$_{n-1} + T_{n-2} + T_{n-3}$	L$_n$
0		0	5	1+2+4	7
1	1	1	6	2+4+7	13
2	0+0+1	1	7	4+7+13	24
3	0+1+1	2	8	7+13+24	44
4	1+1+2	4	9	13+24+44	81

Now, let's take a look when we take modulo 4 of the Tribonacci Number sequence.

$$\{1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \dots\} \to \{(1, 1, 2, 0, 3, 1, 0, 0), 1, 1, 0...\}$$

.

Notice that this sequence repeats after the 8th term, so its period is 8. We can use similar code to what we used for the Fibonacci and Lucas numbers to find the periods of the Tribonacci numbers. Doing that for the first 40 m yields the following table:

m	$\pi_t(m)$	m	$\pi_t(m)$	m	$\pi_t(m)$	m	$\pi_t(m)$	m	$\pi_t(m)$
1	1	9	39	17	96	25	155	33	1430
2	4	10	124	18	156	26	168	34	96
3	13	11	110	19	360	27	117	35	1488
4	8	12	104	20	248	28	48	36	312
5	31	13	168	21	624	29	140	37	469
6	52	14	48	22	220	30	1612	38	360
7	48	15	403	23	553	31	331	39	2184
8	16	16	32	24	208	32	64	40	496

Examining the table, we see that the two conjectures for finding the period of Fibonacci sequence modulo non-prime m also hold true for the Tribonacci numbers. For example, we have that $\pi_T(16) = 32 = 8 \cdot 4 = 8 \cdot \pi_T(2)$. We also have the other conjecture holding true, for example: $\pi_T(20) = 248 = [8, 52] = [\pi_T(4), \pi_T(5)]$. The proof should be the exact same to the proof for Fibonacci.

3. Jacobsthal Numbers

The Jacobsthal numbers are defined by the linear recursion $J_n = J_{n-1} + 2J_{n-2}$, with $J_0 = 0$ and $J_1 = 1$

n	$J_{n-1} + 2J_{n-2}$	J_n	n	$J_{n-1} + 2J_{n-2}$	J_n
0		0	5	2(3)+5	11
1	1	1	6	2(5)+11	21
2	2(0)+1	1	7	2(11)+21	43
3	2(1)+1	3	8	2(21)+43	85
4	2(1)+3	5	9	2(43)+85	171

Now, let's take a look when we take modulo 3 of the Jacobsthal Number sequence.

$$\{1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \dots\} \rightarrow \{(1, 1, 0, 2, 2, 0), 1, 1, 0...\} \quad .$$

Notice that this sequence repeats after the 6th term, so its period is 6. We can use similar code to what we used for the Fibonacci and Lucas numbers to find the periods of the Jabosthal numbers. Doing that for the first 40 m yields the following table:

m	$\pi_j(m)$	m	$\pi_j(m)$	m	$\pi_j(m)$	m	$\pi_j(m)$	m	$\pi_j(m)$
1	1	9	18	17	8	25	20	33	30
2	1	10	4	18	18	26	12	34	8
3	6	11	10	19	18	27	54	35	12
4	2	12	6	20	4	28	6	36	18
5	4	13	12	21	6	29	28	37	36
6	6	14	6	22	10	30	12	38	18
7	6	15	12	23	22	31	10	39	12
8	2	16	2	24	6	32	2	40	4

Examining the table, we see that the conjecture of prime powers doesn't hold true for Jacobsthal numbers, but the conjecture for numbers with multiple prime factors does hold true. For example $\pi_J(14) = 6 = [1, 6] = [\pi_J(2), \pi_J(7)]$. The proof should be the exact same to the proof for Fibonacci.